Lecture note 7: Cost minimization and cost curves

If a firm is maximizing profits and if it chooses to supply some output y, then it must be minimizing the cost of producing y. If this were not so, then there would be some cheaper way of producing y units of output, which would mean that the firm was not maximizing profits in the first place. Therefore profit maximization implies cost minimization.

We will break up the profit maximization problem into two pieces: (1) how to minimize the costs of producing any given amount of output and (2) how to choose the most profitable level of output.

Cost minimization

Suppose we have two factors of production that have prices w_1 and w_2, and that we want to figure out the cheapest way to produce a given level of output, y.

Let x_1 and x_2 measure the amounts used of the two factors and let $f(x_1, x_2)$ be the production function for the firm.

Then cost minimization problem can be written

$$\min_{x_1, x_2} \{w_1 x_1 + w_2 x_2\}$$
$$s.t.\ f(x_1, x_2) = y$$

The solution to this problem will be the minimum costs necessary to achieve the desired level of output given factor prices, i.e.

$$c(w_1, w_2, y)$$

This function is known as **cost function**.

Graphical solution

Isocost line: all the combinations of inputs that have some given level of cost, C, i.e.

$$w_1 x_1 + w_2 x_2 = C$$
$$x_2 = \frac{C}{w_2} - \frac{w_1}{w_2} x_1$$

It is easy to see that this is just a straight line with a slope of $-\frac{w_1}{w_2}$ and a vertical intercept $\frac{C}{w_2}$.

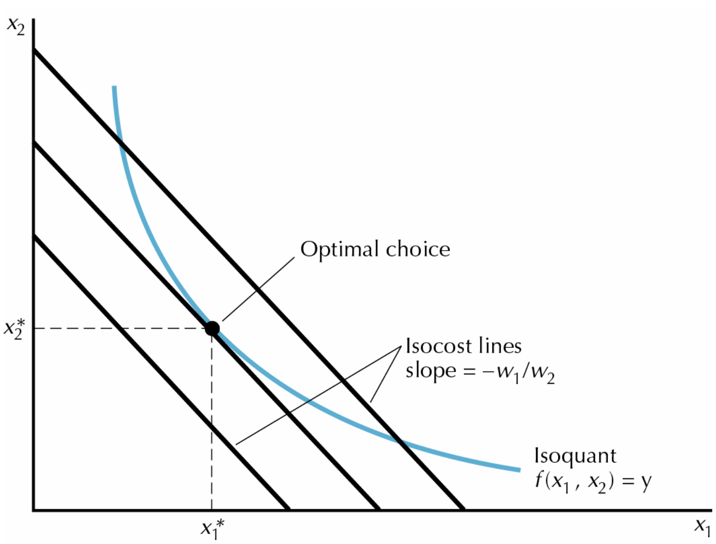


Figure 20.1 Cost minimization

- Every point on an isocost curve has the same cost C.
- Higher isocost lines are associated with higher costs.

Cost minimization problem: find the point on the isoquant that has the lowest possible isocost line associated with it (see figure above).

If the optimal solution involves using some of each factor, and if isoquant is a smooth curve, the cost minimizing point will be characterized by a tangency condition: **the slope of the isoquant must be equal to the slope of the isocost curve**.

Since the slope of the isoquant is TRS and the slope of the isocost line is $-\frac{w_1}{w_2}$, the technical rate of substitution must equal the factor price ratio

$$-\frac{MP_1(x_1^*, x_2^*)}{MP_2(x_1^*, x_2^*)} = TRS(x_1^*, x_2^*) = -\frac{w_1}{w_2}$$

Note that this **tangency condition will not be satisfied** if

- We have a boundary solution where one of the two factors is not used
- The production function has "kinks"

Analytical approach

- Consider any change in the amount of inputs used $(\Delta x_1, \Delta x_2)$ that keeps the output constant

$$MP_1(x_1^*, x_2^*)\Delta x_1 + MP_2(x_1^*, x_2^*)\Delta x_2 = 0$$
$$\Longrightarrow \frac{\Delta x_2}{\Delta x_1} = -\frac{MP_1(x_1^*, x_2^*)}{MP_2(x_1^*, x_2^*)}$$

- If we are at the cost minimum, any change in the amounts of both inputs cannot lower costs, therefore

$$w_1\Delta x_1 + w_2\Delta x_2 \geq 0$$

i.e. if we change that combination of inputs at the cost minimum, costs can only increase. Since this holds for any changes in inputs use this must be true for $(-\Delta x_1, -\Delta x_2)$ as well. Therefore

$$-w_1\Delta x_1 - w_2\Delta x_2 \geq 0$$
$$\Longrightarrow w_1\Delta x_1 + w_2\Delta x_2 \leq 0$$

- The inequalities above can be written together

$$\begin{aligned} w_1\Delta x_1 + w_2\Delta x_2 &\geq 0 \\ w_1\Delta x_1 + w_2\Delta x_2 &\leq 0 \end{aligned}$$

which implies that $w_1\Delta x_1 + w_2\Delta x_2 = 0$. By rearranging

$$\frac{\Delta x_2}{\Delta x_1} = -\frac{w_1}{w_2}$$

From above, we know that

$$\frac{\Delta x_2}{\Delta x_1} = -\frac{MP_1(x_1^*, x_2^*)}{MP_2(x_1^*, x_2^*)}$$

Then

$$-\frac{MP_1(x_1^*, x_2^*)}{MP_2(x_1^*, x_2^*)} = -\frac{w_1}{w_2}$$

which is the condition for cost minimization.

Conditional factor demand functions: the choices of inputs that yield minimal costs for the firm given factor prices and output level, i.e.

$$x_1(w_1, w_2, y) \text{ and } x_2(w_1, w_2, y)$$

Important(!): **conditional** factor demand functions above are different from profit maximizing factor demand functions $x_1(p, w_1, w_2)$ and $x_2(p, w_1, w_2)$.

Example:

- **Perfect complements** factors (fixed proportion technology) $f(x_1, x_2) = \min\{x_1, x_2\}$. If we want to produce y units of output, we clearly need y units of x_1 and y units of x_2. Therefore the minimal cost of production is
$$c(w_1, w_2, y) = w_1 y + w_2 y = (w_1 + w_2) y$$
- **Perfect substitutes** technology $f(x_1, x_2) = x_1 + x_2$. If we want to produce y units of output we would use the cheapest input. Therefore the minimum cost of producing y units of output will be $w_1 y$ or $w_2 y$, whichever is less, i.e.
$$c(w_1, w_2, y) = \min\{w_1 y, w_2 y\} = \min\{w_1, w_2\} y$$

Return to scale and the cost function

Recall the definition of returns to scale, $\forall t > 1$

- **Constant return to scale**: $f(tx_1, tx_2) = tf(x_1, x_2)$
- **Increasing return to scale**: $f(tx_1, tx_2) > tf(x_1, x_2)$
- **Decreasing return to scale**: $f(tx_1, tx_2) < tf(x_1, x_2)$

Consider **CRS case**: suppose that we have solved the cost-minimization problem to produce 1 unit of output, i.e. we know the unit cost function, $c(w_1, w_2, 1)$. What is the cheapest way to produce y units of output?

We just use y times as much of every input as we were using to produce 1 unit of output. This means that the minimal cost to produce y units of output would be

$$c(w_1, w_2, 1) y$$

i.e., **in the case of constant return to scale, the cost function is linear in output**.

Consider **increasing returns to scale case**. If the firm decides to produce twice as much output, it can do so at less than twice the cost, as long as factor prices remain fixed. Intuition: if the firm doubles its inputs, it will more than double its output. Therefore, if it wants to produce double the output, it will be able to do so by using less than twice as much of every input. But using twice as much of every input will exactly double costs. So, using less than twice as much of every input will make costs go up by less than twice. Hence, **the cost function will increase less than linearly with respect to output**.

In **decreasing returns to scale case the cost function will increase more than linearly with respect to output**, i.e. if output doubles, costs will more than double.

Average cost function

The **average cost function** is the cost per unit to produce y units of output

$$AC(y) = \frac{c(w_1, w_2, y)}{y}$$

- **Constant return to scale**

$$\begin{aligned} AC(y) &= \frac{c(w_1, w_2, y)}{y} = \frac{c(w_1, w_2, 1)\, y}{y} \\ &= c(w_1, w_2, 1) \end{aligned}$$

i.e. the cost per unit of output will be constant no matter what level of output the firm wants to produce.

- **Increasing return to scale**: Consider the following numerical example

Technology $y = \min\{x_1^2, x_2^2\}$. It is easy to see that this is increasing return to scale technology. First thing to note is that $x_1 = x_2 = x$ at optimal choice (i.e., the firm does not want to waste the resources). The following table summarizes the input, output and cost levels

y	x_1	x_2	$cost$
1	1	1	$(w_1 + w_2) \cdot 1$
4	2	2	$(w_1 + w_2) \cdot 2$
9	3	3	$(w_1 + w_2) \cdot 3$
...	...	...	...
n	$\sqrt{n}$	$\sqrt{n}$	$(w_1 + w_2) \cdot \sqrt{n}$

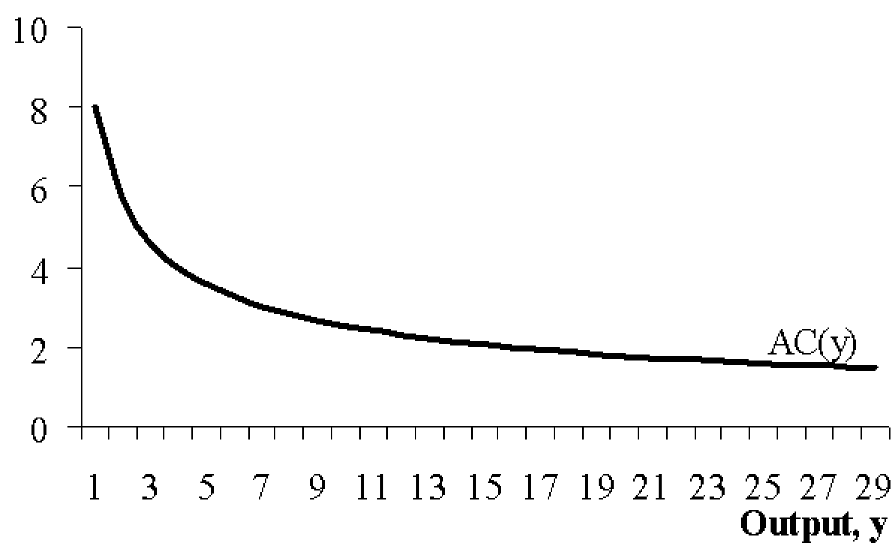
Therefore we can write cost function as follows

$$c(w_1, w_2, y) = (w_1 + w_2) \cdot \sqrt{y}$$

Then average cost is

$$AC(y) = \frac{c(w_1, w_2, y)}{y} = \frac{(w_1 + w_2)}{\sqrt{y}}$$

which looks like (for factor prices $w_1 = 5,\ w_2 = 3$)



That is, the cost will increase less than linearly with respect to output, so the average cost will be declining in output: as output increases, the **average costs of production will tend to fall**.

- **Decreasing return to scale**. Consider the following technology: $y = \min\left\{\sqrt[3]{x_1}, \sqrt[3]{x_2}\right\}$. This is decreasing return to scale technology. Again, $x_1 = x_2 = x$ at optimal choice (i.e., the firm does not want to waste the resources). The following table summarizes the input, output and cost levels

y	x_1	x_2	$cost$
1	1	1	$(w_1 + w_2) \cdot 1$
2	8	8	$(w_1 + w_2) \cdot 8$
3	27	27	$(w_1 + w_2) \cdot 27$
...	...	...	...
n	n^3	n^3	$(w_1 + w_2) \cdot n^3$

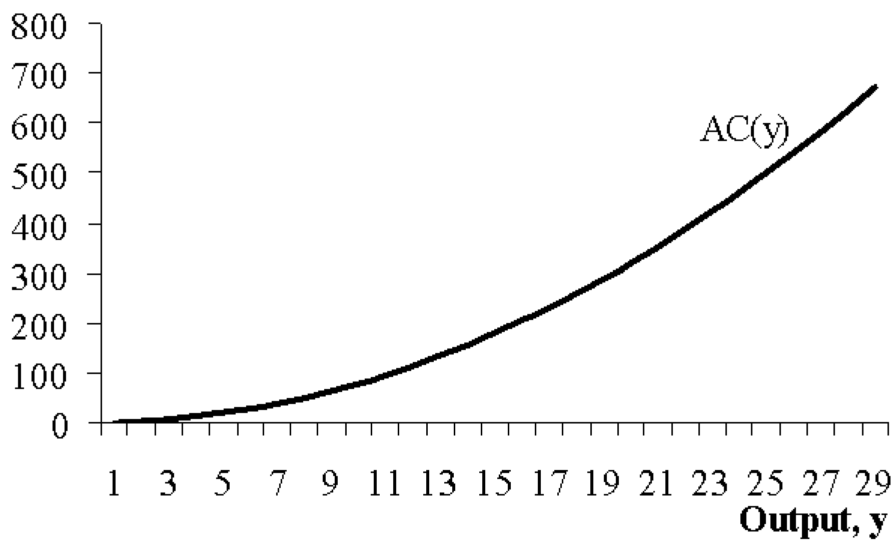
Therefore we can write cost function as follows

$$c(w_1, w_2, y) = (w_1 + w_2) \cdot y^3$$

Then average cost is

$$AC(y) = \frac{c(w_1, w_2, y)}{y} = (w_1 + w_2) \cdot y^2$$

which has the following graph (for factor prices $w_1 = 0.5,\ w_2 = 0.3$)



That is, if the technology exhibits decreasing returns to scale, then average costs will rise as output increases.

Long run and short run costs

Often it is important to distinguish the minimum cost if the firm is allowed to adjust all of its factors of production from the minimum costs if the firm is only allowed to adjust some of its factors.

Short-run cost function is the minimum cost to produce a given level of output, only adjusting the variable factors of production.

$$\begin{aligned} c_s(y, \bar{x}_2) &= \min_{x_1} \{w_1 x_1 + w_2 \bar{x}_2\} \\ s.t. &: f(x_1, \bar{x}_2) = y \end{aligned}$$

with short run factor demands

$$\begin{aligned} x_1 &= x_1^s\left(w_1, w_2, \bar{x}_2, y\right) \\ x_2 &= \bar{x}_2 \end{aligned}$$

Long-run cost function gives the minimum cost of producing a given level of output, adjusting all of the factors of production.

$$\begin{aligned} c\left(y\right) &= \min_{x_1, x_2}\left\{w_1 x_1 + w_2 x_2\right\} \\ s.t. &: f\left(x_1, x_2\right) = y \end{aligned}$$

with long-run factor demands

$$\begin{aligned} x_1 &= x_1\left(w_1, w_2, y\right) \\ x_2 &= x_2\left(w_1, w_2, y\right) \end{aligned}$$

Sunk costs

Sunk costs are another kind of fixed costs: these are costs that have already been incurred and which cannot be recovered.